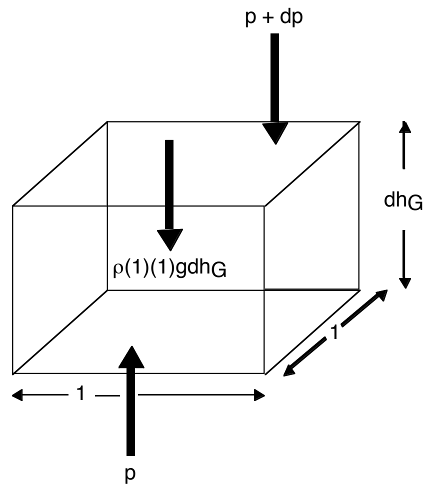
ENAE 283 Equations Derivations

Taken from Introduction to Flight, Fifth Edition *by John D. Anderson, McGraw-Hill, New York, NY, 2005*

The Hydrostatic Equation – Section 3.2, p. 104

Consider an element of fluid that is stationary having sides of unit length and infinitesimally small height dh_G.



Sum the forces:

$$p(1)(1) = (p + dp)(1)(1) + \rho(1)(1)gdh_G$$

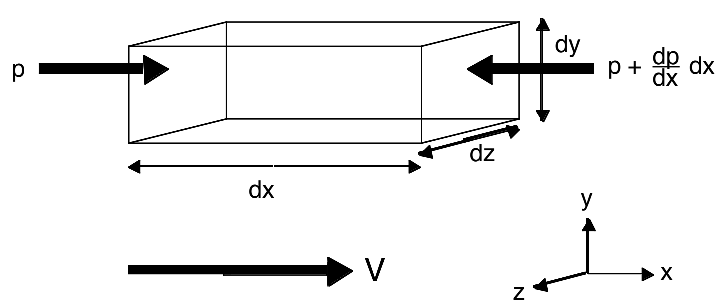
Simplify:

$$p = p + dp + \rho g dh_G$$

$$\boxed{dp = -\rho g dh_G}$$

Euler's Equation – Section 4.3, p. 130

Consider an element of fluid that is moving with velocity V along a streamline. We will neglect frictional and gravitational forces and focus only on the pressures acting on the faces of this element.



Sum the forces:

$$F = pdydz - \left(p + \frac{dp}{dx}dx\right)dydz$$

Simplify:

$$F = -\frac{dp}{dx}(dxdydz)$$

Mass can be found by multiplying the density ρ by the volume *dx dy dz:*

$m = \rho(dx\ dy\ dz)$

Acceleration is the change in velocity divided by the change in time:

$$a = \frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = \frac{dV}{dx}V$$

Using Newton's Second Law: $F = ma$

$$-\frac{dp}{dx}(dxdydz) = \rho(dxdydz)\frac{dV}{dx}V$$

Simplify:

$$\boxed{dp = -\rho V dV}$$

Note: Euler's Equation is valid for inviscid, compressible flow. Gravitational and frictional effects are neglected.

Bernoulli's Equation – Section 4.3, p. 133

Consider incompressible flow along a streamline between points 1 and 2. Euler's Equation can be integrated holding density ρ constant.

$dp = -\rho V dV$	*Euler's Equation*
$\int_{p_1}^{p_2} dp = -\rho \int_{V_1}^{V_2} V dV$	*Integrate between points 1 and 2, holding ρ constant*
$p_2 - p_1 = -\rho\left(\frac{V_2^2}{2} - \frac{V_1^2}{2}\right)$	*Simplify*
$p_1 + \rho\frac{V_1^2}{2} = p_2 + \rho\frac{V_2^2}{2}$	**Bernoulli's Equation**

Note: Bernoulli's Equation is only valid for isentropic, incompressible flow. (And therefore not supersonic flow.)

Isentropic Relations – Section 4.6, p. 147

Consider isentropic flow (adiabatic (no heat added or removed) and reversible) along a streamline.

Apply the first law of thermodynamics with δq=0 (adiabatic and reversible) $\delta q = de + pdv = 0$

Simplify $-pdv = de$

Constant volume process: $de = c_v dT$

I *- Combine:* $-pdv = c_v dT$

Alternate form of the first law: $\delta q = dh + vdp = 0$

Simplify $vdp = dh$

Constant pressure process: $dh = c_p dT$

II *- Combine:* $vdp = c_p dT$

*Divide **I** by **II*** $\frac{-pdv}{vdp} = \frac{c_v}{c_p}$

Simplify $\frac{dp}{p} = -\frac{c_p}{c_v}\frac{dv}{v}$

Define ratio of specific heats (1.4 for air): $\frac{c_p}{c_v} \equiv \gamma$

Simplify $\frac{dp}{p} = -\gamma \frac{dv}{v}$

Integrate between points 1 and 2 $\int_{p_1}^{p_2} \frac{dp}{p} = -\gamma \int_{v_1}^{v_2} \frac{dv}{v}$

Simplify $\ln\frac{p_2}{p_1} = -\gamma \ln\frac{v_2}{v_1}$

Further simplify $\frac{p_2}{p_1} = \left(\frac{v_2}{v_1}\right)^{-\gamma}$

Since $v_1=1/\rho_1$ and $v_2=1/\rho_2$, we can replace v_1 and v_2, remembering to change the sign of γ, as well $\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$

By the equation of state, ρ=p/(RT), $\frac{p_2}{p_1} = \left(\frac{p_2}{RT_2}\frac{RT_1}{p_1}\right)^{\gamma}$

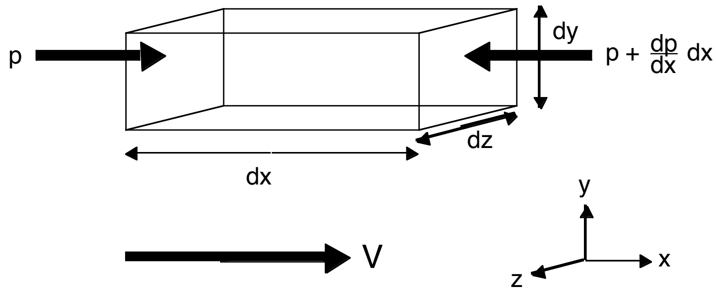
Simplify $\left(\frac{p_2}{p_1}\right)^{1-\gamma} = \left(\frac{T_1}{T_2}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{-\gamma}$

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma(\gamma-1)}$$

Isentropic Relations $\left(\frac{p_2}{p_1}\right) = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}$

Energy Equation – Section 4.7, p. 152

Consider a fluid element moving at velocity V along a streamline:



Apply the first law of thermodynamics	$\delta q + \delta w = de$
Alternate form (v is specific volume with units length³/mass)	$\delta q = dh - vdp$
Since this is an adiabatic flow, δq=0, we can simplify	$dh - vdp = 0$
Euler's Equation	$dp = -\rho V dV$
Combine the last two equations	$dh + v\rho V dV = 0$
Recall that v=1/ρ, so simplify	$dh + VdV = 0$
Integrate between two points on the streamline	$\int_{h_1}^{h_2} dh + \int_{V_1}^{V_2} VdV = 0$
Simplify	$h_1 + \frac{V_1^2}{2} = h2 + \frac{V_2^2}{2}$
Alternate form:	$h + \frac{V^2}{2} = const$
Remember:	$h = c_p T$
Substitute: **Energy Equation**	$c_p T_1 + \frac{1}{2} V_1^2 = c_p T_2 + \frac{1}{2} V_2^2$
Alternate form:	$c_p T + \frac{1}{2} V^2 = const$

Note: The Energy Equation is valid for any flow, including compressible and supersonic conditions, and across shock waves.

Compressible Flow Relations – Section 4.11, p. 174

The definition of enthalpy is $h=e+pv$. Also, $h=c_pT$, $e=c_vT$, and $pv=RT$. Therefore, $c_pT=c_vT+RT$.

Simplify: $$c_p - c_v = R$$

Divide by c_p: $$1-\frac{1}{c_p/c_v}=\frac{R}{c_p}$$

Simplify: $$1-\frac{1}{\gamma}=\frac{\gamma-1}{\gamma}=\frac{R}{c_p}$$

(I) $$c_p=\frac{\gamma R}{\gamma-1}$$

Consider the energy equation relating two points in a flow. One point has conditions T_1 and V_1 and the other is at the stagnation point, with temperature T_0 and zero velocity. The energy equation then becomes:

Energy Equation $$c_pT_1+\frac{1}{2}V_1^2=c_pT_0$$

Rearranged: $$\frac{T_0}{T_1}=1+\frac{V_1^2}{2c_pT_1}$$

Substitute **(I)** *into the last equation:* $$\frac{T_0}{T_1}=1+\frac{V_1^2}{2\left[\gamma R/(\gamma-1)\right]T_1}=1+\frac{\gamma-1}{2}\frac{V_1^2}{\gamma RT_1}$$

Definition of the speed of sound: $$a_1^2=\gamma RT_1$$

Simplify: $$\frac{T_0}{T_1}=1+\frac{\gamma-1}{2}\frac{V_1^2}{a_1^2}$$

Definition of Mach number: $$M_1=V_1/a_1$$

Simplify: $$\frac{T_0}{T_1}=1+\frac{\gamma-1}{2}M_1^2$$

$$\frac{T_0}{T_1}=\left(1+\frac{\gamma-1}{2}M_1^2\right)$$

Compressible Flow Relations: $$\frac{p_0}{p_1}=\left(1+\frac{\gamma-1}{2}M_1^2\right)^{\gamma/(\gamma-1)}$$

$$\frac{\rho_0}{\rho_1}=\left(1+\frac{\gamma-1}{2}M_1^2\right)^{1/(\gamma-1)}$$

Note: These equations are valid for isentropic conditions only, including compressible and supersonic flows.

Area-Velocity Relation – Section 4.13, p. 187

For an isentropic streamline: $\rho_1 A_1 V_1 = \rho_2 A_2 V_2 = const$

Alternatively: $\ln\rho + \ln A + \ln V = \ln(const)$

Differentiate: $\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0$

Recall Euler's Equation: $dp = -\rho V dV$

Rearranged: $\rho = -\frac{dp}{VdV}$

Substitute for ρ: $-\frac{d\rho V dV}{dp} + \frac{dA}{A} + \frac{dV}{V} = 0$

Since the flow is isentropic, we can define: $\frac{d\rho}{dp} = \frac{1}{dp/d\rho} \equiv \frac{1}{a^2}$

Substitute: $-\frac{VdV}{a^2} + \frac{dA}{A} + \frac{dV}{V} = 0$

Rearrange: $\frac{dA}{A} = \frac{VdV}{a^2} - \frac{dV}{V} = \left(\frac{V^2}{a^2} - 1\right)\frac{dV}{V}$

Area-Velocity Relation: $\frac{dA}{A} = \left(M^2 - 1\right)\frac{dV}{V}$

Note: This equation is valid for isentropic conditions only, including compressible and supersonic flows.

Breguet Range Equation – Section 6.12, p. 436

For a propeller-driven aircraft, let *c* be the specific fuel consumption, *P* be the power of the engine, and *dt* be a small increment in time. Thus, *cPdt* will be the weight of the fuel used in an increment of time and thus be the change in the total aircraft weight.

Let W_0 be the total gross weight of the aircraft (full payload and fuel).
Let W_f be the weight of a full fuel load.
Let W_1 be the weight of the plane without fuel.

Thus: $W_1 = W_0 - W_f$

Differentiate: $dW_f = dW = -cPdt$

Rearrange (the negative sign makes sense because *dW* is decreasing with time): $dt = -\frac{dW}{cP}$

Multiply by V_∞ and let *ds* be an incremental change in distance in time *dt*: $V_\infty dt = -\frac{V_\infty dW}{cP} = ds$

Integrate from distance *0* to distance *R* (range): $\int_0^R ds = -\int_{W_0}^{W_1} \frac{V_\infty dW}{cP}$

Rearrange: $R = \int_{W_1}^{W_0} \frac{V_\infty dW}{cP}$

Assume steady conditions where the power available has been adjusted to equal the power required: $P_A = P_R = DV_\infty$

Correct for propeller efficiency ($P_A = \eta P$): $P = \frac{P_A}{\eta} = \frac{DV_\infty}{\eta}$

Substitute and simplify: $R = \int_{W_1}^{W_0} \frac{V_\infty dW}{cP} = \int_{W_1}^{W_0} \frac{V_\infty \eta dW}{V_\infty cD} = \int_{W_1}^{W_0} \frac{\eta dW}{cD}$

Multiply by W / W and recall that L=W for steady, level flight: $R = \int_{W_1}^{W_0} \frac{\eta}{cD}\frac{W}{W} dW = \int_{W_1}^{W_0} \frac{\eta}{c}\frac{L}{D}\frac{dW}{W}$

Rearrange: $R = \frac{\eta}{c}\frac{C_L}{C_D}\int_{W_1}^{W_0} \frac{dW}{W}$

Breguet Range Formula: $R = \frac{\eta}{c}\frac{C_L}{C_D}\ln\frac{W_0}{W_1}$

Breguet Endurance Equation – Section 6.12, p. 436

For a propeller-driven aircraft, let *c* be the specific fuel consumption, *P* be the power of the engine, and *dt* be a small increment in time. Thus, *cPdt* will be the weight of the fuel used in an increment of time and thus be the change in the total aircraft weight.

Let W_0 be the total gross weight of the aircraft (full payload and fuel).
Let W_f be the weight of a full fuel load.
Let W_1 be the weight of the plane without fuel.

Thus: $$W_1 = W_0 - W_f$$

Differentiate: $$dW_f = dW = -cPdt$$

Rearrange (the negative sign makes sense because *dW* is decreasing with time): $$dt = -\frac{dW}{cP}$$

Integrate from *0* to time *E* (empty): $$\int_0^E dt = -\int_{W_0}^{W_1} \frac{dW}{cP}$$

Recall that P=DV$_\infty$/η and that W=L: $$E = \int_{W_1}^{W_0} \frac{dW}{cP} = \int_{W_1}^{W_0} \frac{\eta}{c}\frac{dW}{DV_\infty} = \int_{W_1}^{W_0} \frac{\eta}{c}\frac{L}{DV_\infty}\frac{dW}{W}$$

For steady, level flight: $$L = W = \tfrac{1}{2}\rho_\infty V_\infty^2 S C_L$$

Solve for V$_\infty$: $$V_\infty = \sqrt{2W/(\rho_\infty S C_L)}$$

Substitute: $$E = \int_{W_1}^{W_0} \frac{\eta}{c}\frac{C_L}{C_D}\sqrt{\frac{\rho_\infty S C_L}{2}}\frac{dW}{W^{3/2}}$$

Simplify, remembering that C_L, C_D, η, c, and ρ_∞ are constant: $$E = -2\frac{\eta}{c}\frac{C_L^{3/2}}{C_D}\left(\frac{\rho_\infty S}{2}\right)^{1/2}\left[W^{-1/2}\right]_{W_1}^{W_0}$$

Breguet Endurance Formula: $$E = \frac{\eta}{c}\frac{C_L^{3/2}}{C_D}(2\rho_\infty S)^{1/2}\left(W_1^{-1/2} - W_0^{-1/2}\right)$$